

Thermodynamics of Near-Extreme Black Holes

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Abstract

The thermodynamics of nearly-extreme charged black holes depends upon the number of ground states at fixed large charge and upon the distribution of excited energy states. Here three possibilities are examined: (1) Ground state highly degenerate (as suggested by the large semiclassical Hawking entropy of an extreme Reissner-Nordstrom black hole), excited states not. (2) All energy levels highly degenerate, with macroscopic energy gaps between them. (3) All states nondegenerate (or with low degeneracy), separated by exponentially tiny energy gaps. I suggest that in our world with broken supersymmetry, this last possibility seems most plausible. An experiment is proposed to distinguish between these possibilities, but it would take a time that is here calculated to be more than about 10^{837} years.

1 Introduction

What are the energy levels of a charged black hole? For fixed electric charge $Q \gg 1$, the minimum mass of a classical electrically charged black hole (given by the Kerr-Newman metric, or, in the nonrotating case, by the Reissner-Nordstrom metric) is $M = Q$. In this paper, we shall examine the consequences of various assumption for the levels of the excess energy

$$E \equiv M - Q. \quad (1)$$

(Strictly speaking, vacuum polarization is likely to shift the minimum mass for fixed charge Q slightly away from Q if the theory does not have unbroken supersymmetry, in which case in Eq. (1), and in many equations below, one should replace Q by the minimum for the mass M for the fixed charge. However, for simplicity I shall here and henceforth simply call this minimum mass Q .)

At $E = 0$, one has (classically) an extreme-charged Reissner-Nordstrom black hole with event horizon radius $r_+ = M = Q$ and hence Bekenstein-Hawking entropy (using Planck units throughout, $\hbar = c = G = k = 4\pi\epsilon_0 = 1$)

$$S_0 \equiv S_H(E = 0) = A/4 = \pi Q^2 \gg 1. \quad (2)$$

This formula for the entropy of an extreme black hole is also supported by D -brane calculations for black holes in superstring theory with unbroken supersymmetry. It suggests that the ground state of a black hole with large fixed charge is highly degenerate. On the other hand, there have been counter-arguments [1, 2] suggesting that the entropy of an extreme black hole is zero. Which idea is correct?

There seem to be three extreme possibilities in our world with broken supersymmetry, though of course intermediate possibilities also exist: (1) Ground state highly degenerate, excited states not. (2) All states highly degenerate, with macroscopic mass gaps. (3) All states nondegenerate, separately microscopically. Here I wish to examine some of the thermodynamic consequences of these possibilities.

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2 Semiclassical Thermodynamics

For classical Kerr-Newman black holes with given Q and E (or $M = Q + E$), the maximum Bekenstein-Hawking entropy is obtained if the black holes have zero angular momentum and are hence Reissner-Nordstrom black holes. Therefore, states with zero (or at least relatively small) angular momentum might be expected to dominate in number for a given Q and range of E , and so we shall focus on such black holes.

The classical Reissner-Nordstrom metric is

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2d\Omega^2, \quad (3)$$

where $d\Omega^2$ is the metric on the unit S^2 and where

$$\Delta \equiv r^2 - 2Mr + Q^2 \equiv (r - r_+)(r - r_-), \quad (4)$$

with

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \quad (5)$$

being the radii of the two horizons ($r = r_+$ at the outer or event horizon, and $r = r_-$ at the inner or Cauchy horizon).

In terms of r_+ and r_- , or in terms of the event horizon radius r_+ and the charge Q , which are two convenient parameters for describing the Reissner-Nordstrom metrics, the black hole mass is

$$M = \frac{1}{2}(r_+ + r_-) = \frac{1}{2}r_+ + \frac{Q^2}{2r_+}, \quad (6)$$

and so the energy excess over the minimum for that Q is

$$E \equiv M - Q = \frac{(r_+ - Q)^2}{2r_+}. \quad (7)$$

The Bekenstein-Hawking entropy is

$$\begin{aligned} S &= \frac{1}{4}A = \pi r_+^2 = \pi(M + \sqrt{M^2 - Q^2})^2 \\ &= \pi(Q + E + \sqrt{2QE + E^2})^2 \\ &\approx \begin{cases} \pi Q^2 + \pi\sqrt{8Q^3E} & \text{for } E \ll Q \\ 4\pi E^2 & \text{for } E \gg Q \end{cases} \end{aligned} \quad (8)$$

From these relations, one can easily derive that the Hawking temperature is

$$\begin{aligned} T_H &= \left(\frac{\partial E}{\partial S} \right)_Q = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{r_+^2 - Q^2}{4\pi r_+^3} = \frac{Mr_+ - Q^2}{2\pi r_+^3} \\ &\approx \begin{cases} \sqrt{\frac{E}{2\pi^2 Q^3}} & \text{for } E \ll Q \Rightarrow T_H \sim E \text{ for } E \sim \frac{1}{Q^3} \\ \frac{1}{8\pi E} & \text{for } E \gg Q \end{cases} \end{aligned} \quad (9)$$

and that the heat capacity (at fixed charge) is

$$C_Q = \left(\frac{\partial E}{\partial T_H} \right)_Q = \frac{2\pi r_+^2(r_+^2 - Q^2)}{3Q^2 - r_+^2}. \quad (10)$$

The heat capacity is thus positive for $r_+ < \sqrt{3}Q$ or $M < 2Q/\sqrt{3}$ or $E < Q/(2\sqrt{3} + 3) \approx 0.1547Q$.

The free energy of a Reissner-Nordstrom black hole is

$$F = E - TS = \frac{(r_+ - Q)^2}{2r_+} - \pi T r_+^2, \quad (11)$$

which has no global minimum, as a function of r_+ (or of E), for fixed Q and T . Therefore, even though black holes with $C_Q > 0$ are locally stable against small energy increases or decreases from absorbing or emitting small amounts of thermal radiation, they are globally unstable to absorbing an arbitrarily large amount of radiation from the hypothetical infinite heat bath. This is a case of the general result that in asymptotically flat spacetime with an infinitely large heat bath at infinity, any black hole is unstable to growing indefinitely with negative heat capacity (once the energy gets sufficiently larger than the charge) and entropy $S \sim 4\pi E^2$. In other words, the canonical ensemble, weighting all states by $e^{-E/T}$, is divergent and hence not defined for gravitational systems in asymptotically flat spacetime, with T being the temperature at asymptotic infinity (the only place where one can have an infinite heat bath) [3].

To define a canonical ensemble artificially, one may introduce a negative cosmological constant, $\Lambda \equiv -3/b^2 < 0$ [4]. Then the metric for a charged black hole with zero angular momentum has the same form as Eq. (3), but now with

$$\Delta \equiv r^2 - 2Mr + Q^2 + r_+^4/b^2. \quad (12)$$

The explicit formula for the event horizon radius r_+ , the greatest radius where Δ vanishes, now involves the solution of a quartic polynomial and hence is rather complicated when written in terms of M , Q , and b (or Λ), but the explicit expressions for various quantities remain simple when written in terms of r_+ and Q (and in terms of b , which is assumed to be kept fixed as part of the theory). In particular,

$$M = \frac{1}{2}r_+ + \frac{Q^2}{2r_+} + \frac{r_+^3}{2b^2}, \quad (13)$$

and

$$S = \pi r_+^2 \sim \pi(2b^2 M)^{2/3} \text{ for } M \gg b, \quad (14)$$

from which one may derive that

$$T_H = \left(\frac{\partial M}{\partial S} \right)_Q = \frac{1}{4\pi r_+} \left(\frac{3r_+^2}{b^2} + 1 - \frac{Q^2}{r_+^2} \right) \quad (15)$$

and

$$C_Q = \left(\frac{\partial M}{\partial T_H} \right)_Q = 2\pi r_+^2 \left(\frac{3r_+^4 + b^2 r_+^2 - b^2 Q^2}{3r_+^4 - b^2 r_+^2 + 3b^2 Q^2} \right). \quad (16)$$

The fact that the entropy does not rise so fast as linearly with the mass for $M \gg b$ means that with the negative cosmological constant, the canonical ensemble is well defined [4], as opposed to the asymptotically flat case (zero cosmological constant) in which the entropy rises faster than linearly, $S \sim 4\pi M^2$. Alternatively, the free energy (using M in place of E , since otherwise the expression becomes rather unwieldy) is

$$F = M - TS = \frac{r_+^3}{2b^2} - \pi T r_+^2 + \frac{r_+}{2} + \frac{Q^2}{2r_+}, \quad (17)$$

which is now bounded below for fixed Q , T , and b , having one or two minima.

There are two minima if $b > 6Q$, and indeed we shall be interested in the case that $b \gg Q$, so that the negative cosmological constant gives only a tiny perturbation of the metric near the black hole. Then if (but not only if) $b < 1/(\pi T)$, the minimum at least positive r_+ is the global minimum. For $Q \ll b < 1/(\pi T)$, this global minimum occurs when T is the Hawking temperature of the black hole (with T defined with respect to a timelike Killing vector that is normalized to have unit magnitude at the location where $g_{00} = -1$),

$$T = T_H = \frac{1}{4\pi r_+} \left(\frac{3r_+^2}{b^2} + 1 - \frac{Q^2}{r_+^2} \right), \quad (18)$$

which is very nearly that given by Eq. (9) for a Reissner-Nordstrom black hole with zero cosmological constant, obtained by taking b to infinity in Eq. (18).

For example, the minimum that gives $E \sim T_H \sim 1/Q^3 \ll 1$ is a global minimum of the free energy F if $b \lesssim Q^3$, which for $Q \gg 1$ (as we are assuming) allows the length scale b of the cosmological constant to be much larger than the length scale $r_+ \approx Q$ of the nearly extreme black hole. To take a value of Q that arises

below, $Q \sim 3 \times 10^{46} \sim 4 \times 10^8 M_\odot$ gives the very weak restriction $b \lesssim Q^3 \sim 3 \times 10^{82}$ Mpc in order for this configuration to be a global minimum of the free energy. Therefore, for $T \ll 1/Q$, take $b \sim 1/(\pi T) \gg Q$ to stabilize the canonical ensemble but then ignore correction terms like $r_+^2/b^2 \sim Q^2 T^2 \ll 1$ from the tiny cosmological constant.

Some strange thermodynamic behavior occurs when $E \lesssim T$. Preskill *et al.* [5] argued that the thermal description of a black hole breaks down when the emission of a particle of energy T changes T by a significant fraction of itself. For a charged Reissner-Nordstrom black hole with $E = M - Q \ll Q$, this occurs for $E \lesssim 1/Q^3$ or $T \lesssim 1/Q^3$, where $E \lesssim T$. Such a black hole does not have sufficient energy to emit a particle of energy T .

Another argument for strange thermodynamic behavior at $E \lesssim 1/Q^3$ came from a derivation by Maldacena and Susskind [6] of a mass gap $E_{\text{gap}} = 1/(8Q^3)$ from string theory. Maldacena and Strominger [7] showed that it could be derived semiclassically by implicitly assuming that the first excited state of an extreme Reissner-Nordstrom black hole is an extreme Kerr-Newman black hole with angular momentum $J = 1/2$:

$$E \equiv M - Q = \sqrt{Q^2 + a^2} - Q \approx \frac{a^2}{2Q} = \frac{J^2}{2QM^2} \approx \frac{J^2}{2Q^3} = \frac{1}{8Q^3}. \quad (19)$$

Of course, this derivation fails if the lowest excited states have $J = 0$, as would occur for an ordinary chunk of solid in the idealization of zero baryon decay.

3 Evolving a Black Hole to Energy $E \sim 1/Q^3$

To get a black hole to $E = M - Q \lesssim 1/Q^3$ by emitting its excess energy, we need it to have a charge Q so huge that it Hawking radiates to this point without emitting charge. Assume that only photons and gravitons are massless, and that all other particles (e.g., neutrinos) have rest masses much larger than the temperature of the hole. Then the primary emission will be to photons with unit angular momentum. The three possible values of the axial angular momentum, and the two helicities for each, then give an energy emission rate

$$\frac{dM}{dt} \approx -\frac{6}{2\pi} \int_0^\infty \frac{P\omega d\omega}{e^{\omega/T} - 1}. \quad (20)$$

Here $P = P(Q/M, M\omega)$ is the absorption probability for a photon of frequency ω and unit angular momentum by a Reissner-Nordstrom black hole of mass M and charge Q .

This absorption probability can be obtained from the solution to the radial equation [8, 9]

$$\left(\frac{d^2}{dr_*^2} + \omega^2 - \frac{2\Delta}{r^4} - \frac{4Q^2\Delta}{r^6} \right) Z(r) = 0 \quad (21)$$

obtained after separating variables for the photon mode, where

$$\frac{d}{dr_*} = \frac{\Delta}{r^2} \frac{d}{dr} = \frac{r^2 - 2Mr + Q^2}{r^2} \frac{d}{dr}. \quad (22)$$

In particular, the boundary conditions for the radial mode function $Z(r)$ are

$$r_* \rightarrow +\infty: \quad Z \sim Z_{\text{in}} e^{-i\omega r_*} + Z_{\text{out}} e^{i\omega r_*}, \quad (23)$$

$$r_* \rightarrow -\infty: \quad Z \sim Z_{\text{down}} e^{-i\omega r_*}. \quad (24)$$

Then the absorption probability is

$$P = \left| \frac{Z_{\text{down}}}{Z_{\text{in}}} \right|^2 = 1 - \left| \frac{Z_{\text{out}}}{Z_{\text{in}}} \right|^2. \quad (25)$$

Then the goal is to evaluate $P(Q/M, M\omega)$ from Eq. (25) and use it in Eq. (20) to find the energy emission rate and thereby find a minimum charge Q_* such that if and only if $Q > Q_*$, the time to emit energy in uncharged particles (mainly photons) to go to $E = 1/(8Q^3) \ll 1$ is less than the time to emit

a positron. Even a single positron emission would spoil the process of getting to $E = 1/(8Q^3) \ll 1$, since it would reduce the black hole charge Q by the positron charge $e = \sqrt{\alpha} \approx 0.085424543$, and reduce the black hole mass M by the much smaller amount of roughly the positron mass, $m \approx 4.18 \times 10^{-23}$ in the Planck units we are using, so that during the positron emission, $E \equiv M - Q$ would *increase* by roughly $0.08542 \gg 1/(8Q^3)$.

First, we look for an approximate solution to the radial Eq. (21) for the frequencies that dominate the thermal emission for a nearly extreme black hole. These are frequencies ω that have roughly the same order of magnitude as the Hawking temperature T_H . Let

$$\epsilon \equiv \frac{r_+ - r_-}{r_+} \ll 1 \quad (26)$$

be the tiny quantity that parametrizes how close to extremality the black hole is. Then

$$T_H = \frac{\epsilon}{4\pi r_+}, \quad (27)$$

where, as discussed above, we are here and henceforth neglecting the effect of the miniscule cosmological constant needed to stabilize the canonical ensemble, and we are also neglecting the effect of the angular momentum of the hole, which is also expected to be relatively small for most states of fixed Q for $E = M - Q \ll Q$. Therefore, we are considering just pure Reissner-Nordstrom black holes that are very near their critical charge.

Eq. (27) implies that for frequencies ω that have roughly the same order of magnitude as the Hawking temperature T_H , the quantity $r_+\omega$ is of the order of ϵ and hence is very small. Therefore, we shall solve the radial Eq. (21) assuming both $\epsilon \ll 1$ and $r_+\omega \ll 1$. This then allows

$$r_+\omega \ll \nu \equiv \frac{r_+\omega}{\epsilon} = \frac{\omega}{4\pi T_H} \ll \frac{1}{\epsilon}. \quad (28)$$

That is, the dimensionless frequency factor ν (or $4\pi\nu = \omega/T_H$) that occurs in the thermal Hawking emission formula can range from zero (when $r_+\omega = 0$) to a very large upper limit (much smaller than, but within a few orders of magnitude of, $1/\epsilon$), thereby spanning effectively all of the thermal emission range to give an excellent approximation to the mass emission rate given by Eq. (20).

To solve Eq. (21) using the approximations of Eq. (28), split the region external to the black hole horizon into three overlapping regions, solve Eq. (21) in each region, and match in the two overlap regions.

In the near-horizon Region I defined by $r - r_+ \ll r_+$,

$$Z \approx Z_{\text{down}} \frac{r}{r_+} \left(\frac{r - r_-}{r_+ - r_-} \right)^2 \left(\frac{r - r_+}{r - r_-} \right)^{-i\nu} \left[1 + \frac{4(1+i\nu)}{1-2i\nu} \left(\frac{r - r_+}{r - r_-} \right) + \frac{(1+i\nu)(1+2i\nu)}{(1-i\nu)(1-2i\nu)} \left(\frac{r - r_+}{r - r_-} \right)^2 \right]. \quad (29)$$

In the intermediate Region II defined by $r_- - r_- \ll r - r_+ \ll 1/\omega$,

$$Z \approx A \frac{(r + 2r_+)(r - r_+)^2}{r} + B \frac{r^4(r - 3r_+)}{(r - r_+)^3}. \quad (30)$$

And in the far-field Region III defined by $r_+ \ll r$,

$$Z \sim Z_{\text{in}} e^{-i\omega r_*} \left(1 - \frac{i}{\omega r_+} \right) + Z_{\text{out}} e^{i\omega r_*} \left(1 + \frac{i}{\omega r_+} \right). \quad (31)$$

Then we match the solutions across Regions I and II for $r_+ - r_- \ll r - r_+ \ll r_+$, using $\epsilon \ll 1$ to guarantee the existence of this overlap region, to determine the constants A and B of Eq. (30) in terms of Z_{down} . Similarly, we match the solutions across Regions II and III for $r_+ \ll r \ll 1/\omega$, using $r_+\omega \ll 1$ to guarantee the existence of this overlap region, to determine the constants Z_{in} and Z_{out} of Eq. (31) in terms of A and B (and hence in terms of Z_{down} from the results of the previous overlap calculation).

The result is the absorption probability for photons of unit angular momentum over the entire dominant part of the Hawking emission spectrum:

$$\begin{aligned} P = \left| \frac{Z_{\text{down}}}{Z_{\text{in}}} \right|^2 &\approx \frac{1}{9} \epsilon^8 \nu^4 (1 + \nu^2) (1 + 4\nu^2) \\ &= \frac{1}{9} r_+^4 \omega^4 (\epsilon^2 + r_+^2 \omega^2) (\epsilon^2 + 4r_+^2 \omega^2). \end{aligned} \quad (32)$$

Again, this approximate result is valid for $\epsilon = (r_+ - r_-)/r_+ \ll 1$ and for $\nu \ll 1/\epsilon$ or $\omega \ll 1/r_+ = 4\pi T_H/\epsilon$, and within this approximation, the emission of unit angular momentum photons overwhelmingly dominates over the emission of higher angular momentum photons and gravitons, by several factors of the large number $1/\epsilon$.

One can use this result to determine that the absorption cross section of a nearly-extreme Reissner-Nordstrom black hole to photons with $r_+ \omega \ll 1$ is approximately

$$\sigma = \frac{(2\ell + 1)\pi}{\omega^2} P \approx \frac{\pi}{3} r_+^4 \omega^2 (\epsilon^2 + r_+^2 \omega^2) (\epsilon^2 + 4r_+^2 \omega^2). \quad (33)$$

It is interesting that although this formula was derived only for nearly-extreme black holes, $\epsilon \ll 1$, it gives the correct low-frequency limit for the uncharged Schwarzschild metric, which has $\epsilon = 1$, namely $\sigma \approx (16\pi/3)M^4 \omega^2$. On the other hand, for a precisely extreme Reissner-Nordstrom black hole, $\epsilon = 0$ and $r_+ = Q$, we get that the low-frequency photons ($Q\omega \ll 1$) have $\sigma \approx (4\pi/3)Q^8 \omega^6$.

Now we can insert Eq. (32) into Eq. (20) and integrate to determine the approximate time for a black hole with sufficiently large charge to evolve to $E \equiv M - Q = 1/(8Q^3) \ll 1$. In particular, if the charge is sufficiently large that $|dQ/dt| \ll |dM/dt|$ (and how large is sufficient will be discussed below), and if $E \equiv M - Q \ll Q$, then $\epsilon \approx \sqrt{8E/Q}$ and $\omega \approx \epsilon\nu/Q \approx \nu\sqrt{8E/Q^3}$, so

$$\begin{aligned} \frac{dE}{dt} &\approx \frac{dM}{dt} \approx -\frac{6}{2\pi} \int_0^\infty \frac{P\omega d\omega}{e^{\omega/T} - 1} = -\frac{3\epsilon^2}{\pi Q^2} \int_0^\infty \frac{P\nu d\nu}{e^{4\pi\nu} - 1} \\ &\approx -\frac{\epsilon^{10}}{3\pi Q^2} \int_0^\infty \frac{(\nu^5 + 5\nu^7 + 4\nu^9)d\nu}{e^{4\pi\nu} - 1} = -\frac{509\epsilon^{10}}{42\,577\,920\pi Q^2} \approx -\frac{4\,072\,E^5}{10\,395\pi Q^7}. \end{aligned} \quad (34)$$

The time to Hawking radiate to $E = 1/(8Q^3)$ is then

$$t \approx \frac{10\,395\pi Q^7}{16\,288\,E^4} \approx \frac{1\,330\,560}{509}\pi Q^{19} \approx 8\,212.333\,047\,46\,Q^{19}. \quad (35)$$

Next, we need to find how large Q needs to be, $Q \gtrsim Q_*$, in order that the probability be low that the charged black hole will emit a positron during this time and hence spoil the process of slowly evolving to $E = 1/(8Q^3)$. For a black hole with $1 \ll Mm \ll eQ \ll (Mm)^2$, where m and e are the mass and charge of a positron, the Hawking emission rate of positrons is well approximated by integrating the Schwinger pair production rate over the exterior of the black hole, where the electric field strength is $\mathcal{E} = Q/r^2$:

$$\begin{aligned} \frac{dN}{dt} &\approx \int_{r_+}^\infty 4\pi r^2 dr \frac{e^2 \mathcal{E}^2}{4\pi^3} \sum_{n=1}^\infty \frac{1}{n^2} \exp\left(-\frac{n\pi m^2}{e\mathcal{E}}\right) \\ &\approx \int_{r_+}^\infty \frac{e^2 Q^2}{\pi^2 r^2} dr \exp\left(-\frac{\pi m^2 r^2}{eQ}\right) \\ &\approx \frac{e^3 Q^3}{2\pi^3 m^2 r_+^3} \exp\left(-\frac{r_+^2}{Q_0 Q}\right) \approx \frac{e^3}{2\pi^3 m^2} \exp\left(-\frac{Q}{Q_0}\right), \end{aligned} \quad (36)$$

where the last approximate equality assumes $\epsilon \ll 1$ so that $r_+ \approx Q$, and where

$$\begin{aligned} Q_0 &\equiv \frac{e}{\pi m^2} = \frac{em_p^2}{\pi m^2} m_p^{-2} \approx 1.553 \times 10^{43} \\ &\approx 1.8 \times 10^{44} e \approx 1.7 \times M_\odot \approx 0.84 \text{ s}. \end{aligned} \quad (37)$$

We need that this positron emission rate be less than the reciprocal of the time to Hawking radiate to $E = 1/(8Q^3)$ that is given by Eq. (35). This implies that we need

$$Q > Q_* = Q_0 x \quad (38)$$

with x the solution of

$$x - 19 \ln x = y \equiv \ln \left(\frac{665 \, 280 \, e^{22}}{509 \, \pi^{21} m^{40}} \right) \approx 1990. \quad (39)$$

This implies that

$$x \approx y + 19 \ln y + \frac{19^2}{y} \ln y \approx 2136. \quad (40)$$

Then one gets the minimum charge Q_* of a black hole that would Hawking radiate to $E = 1/(8Q^3)$ before it is likely to emit a positron and thereby have $E \equiv M - Q$ jump upward by much more than $1/(8Q^3)$ from the reduction of Q by the emitted positron charge e :

$$\begin{aligned} Q_* &\approx 3.316 \times 10^{46} \approx 3.88 \times 10^{47} e \approx 3.6 \times 10^8 M_\odot \approx 3.6 \text{ AU} \\ &\approx 5.4 \times 10^{11} \text{ meters} \approx 1788 \text{ seconds} \approx 0.497 \text{ hour}. \end{aligned} \quad (41)$$

The magnitude of this last quantity, applied for the size of a silent object that might exist in the heavens, reminded me of a verse in the last book of the Bible, Revelation 8:1: “When the Lamb opened the seventh seal, there was silence in heaven for about half an hour.”

Now if we go back and calculate the times given by Eq. (35) and by the reciprocal of the rate given by Eq. (36), we find that for $Q > Q_*$ the time to Hawking radiate to $E = 1/(8Q^3)$ is

$$t \approx (1.1 \times 10^{837} \text{ yr}) \left(\frac{Q}{Q_*} \right)^{19}, \quad (42)$$

which is less than the expected time to emit a positron,

$$t \approx (1.1 \times 10^{837} \text{ yr}) \exp \left[\frac{2136(Q - Q_*)}{Q_*} \right]. \quad (43)$$

Q_* is the charge of 6.5×10^{20} kg of protons, 0.88% of the lunar protons, or 65% of the asteroid Ceres' protons. If superclusters of mass $M_S \sim 10^{16} M_\odot$ have charge $Q_S \sim m_p M_S / e \sim 9.00 \times 10^{-19} M_S \sim 8 \times 10^{35} \sim 2.5 \times 10^{-11} Q_*$, we would need $\sim 4 \times 10^{10}$ such superclusters to collapse into the black hole without the surrounding negative plasma to get a black hole of charge Q_* . If a sphere of N_S such superclusters collapses, with the $\sim N_S^{2/3}$ on the surface falling in unneutralized, one needs $N_S \sim (4 \times 10^{10})^{3/2} = 8 \times 10^{15}$ such superclusters. This would initially form a black hole of mass $M \sim N_S M_S \sim 10^{32} M_\odot \sim 10^{70} \sim 10^{19}$ yr, with initial charge-to-mass ratio $Q/M \sim 10^{-23}$.

Using the results of [10] for the emission of photons and gravitons (here assumed to be the only particles of mass less than $1/M \sim 10^{-70} \sim 10^{-42} eV$, as discussed above) from a nearly Schwarzschild black hole, the time to evaporate to the point where the charge becomes important for the geometry and makes the Hawking temperature attain its maximum value of $1/(6\sqrt{3}\pi Q_*) \approx 9.24 \times 10^{-49} \approx 1.31 \times 10^{-16} K \approx 1.13 \times 10^{-20} eV$ is roughly $8895 M^3 \sim 10^{163}$ years. Then the time to evaporate to $E = 1/(8Q^3) = 1/(8Q_*^3) \approx 3.43 \times 10^{-141} \approx 4.19 \times 10^{-113} eV$ (as the temperature plummets to $1/(4\pi Q_*^3) \approx 2.18 \times 10^{-141} \approx 3.09 \times 10^{-109} K$) would be roughly the 10^{837} years discussed above. One can see that even if the universe expands as slowly as a matter-dominated $k = 0$ Friedmann-Robertson-Walker universe, with scale size going as the time to the 2/3 power, the current $3K$ microwave background would have by then cooled below $10^{-551} K$ and so would be negligible in comparison with the roughly $10^{-109} K$ temperature of the nearly extreme black hole.

4 Density of States

Let us consider various possibilities for the density of states (energy levels) of a nearly extreme charged black hole. First, let us make a semiclassical estimate. As we found above for nearly extreme black holes,

$E \equiv M - Q \ll Q$ (neglecting the effect of the tiny cosmological constant that was introduced to stabilize the canonical ensemble), we get that the Hawking temperature T_H (here denoted simply as T) is

$$T = \frac{r_+ - r_-}{4\pi r_+^2} \approx \sqrt{\frac{E}{2\pi^2 Q^3}}, \quad (44)$$

and the entropy is

$$S = \pi r_+^2 \approx \pi Q^2 + \sqrt{8\pi^2 Q^3 E}. \quad (45)$$

Expressed in terms of the temperature, this gives

$$E \approx 2\pi^2 Q^3 T^2 \quad (46)$$

and

$$S \approx \pi Q^2 + 4\pi^2 Q^3 T. \quad (47)$$

If E is reinterpreted to be $\langle E \rangle = -d \ln Z / d\beta$ with $\beta = 1/T$ and partition function

$$Z(\beta) = \sum_{n=1}^{\infty} e^{-\beta E_n} \quad (48)$$

from energy levels E_n , then

$$\ln Z = S - \beta \langle E \rangle \approx \pi Q^2 + 2\pi^2 Q^3 \beta^{-1} \quad (49)$$

or

$$Z \approx e^{\pi Q^2 + 2\pi^2 Q^3 T}. \quad (50)$$

The inverse Laplace transform gives the density of states

$$\rho(E) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(\beta) e^{\beta E} d\beta \approx e^{\pi Q^2} \delta(E) + e^{\pi Q^2} \sqrt{\frac{2\pi^2 Q^3}{E}} I_1 \left(\sqrt{8\pi^2 Q^3 E} \right) \theta(E), \quad (51)$$

with I_1 (and I_0 below) being Bessel functions of imaginary argument. The number of states $\leq E$ for $E \geq 0$ is then

$$\begin{aligned} N(E) &= \int_{-\infty}^E \rho(E') dE' \approx e^{\pi Q^2} I_0 \left(\sqrt{8\pi^2 Q^3 E} \right) \\ &= e^{\pi Q^2} \left[1 + \frac{2\pi^2 Q^3 E}{(1!)^2} + \frac{(2\pi^2 Q^3 E)^2}{(2!)^2} + \dots \right]. \end{aligned} \quad (52)$$

For $1/Q^3 \ll E \ll Q$,

$$N(E) \approx \frac{e^{\pi Q^2 + \sqrt{8\pi^2 Q^3 E}}}{2\pi(2Q^3 E)^{1/4}} \approx \frac{e^{S(E)}}{2\pi(2Q^3 E)^{1/4}}. \quad (53)$$

If $S_0 \equiv S(E=0) = \pi Q^2$, perhaps a good approximation for all $E \geq 0$ is that the number of states of lower E is, by this semiclassical estimate,

$$\begin{aligned} N(E) &\approx e^{S_0} I_0[S(E) - S_0] = e^{\pi Q^2} I_0[\pi r_+^2(E) - \pi Q^2] \\ &= e^{\pi Q^2} I_0[\pi(Q + E + \sqrt{2QE + E^2})^2 - \pi Q^2]. \end{aligned} \quad (54)$$

This gives a huge $e^{\pi Q^2}$ degeneracy of states at $E=0$ and then a very dense quasi-continuum of states. For $E \ll 1/(2\pi^2 Q^3)$, $N(E) \approx e^{\pi Q^2} (1 + 2\pi^2 Q^3 E)$, giving a density of states

$$\rho(E) = \frac{dN(E)}{dE} \approx e^{\pi Q^2} [\delta(E) + 2\pi^2 Q^3 \theta(E)]. \quad (55)$$

The quasi-continuum of excited states has an average energy separation

$$\delta E = \frac{1}{\rho} \approx \frac{e^{-\pi Q^2}}{2\pi^2 Q^3}, \quad (56)$$

exponentially small.

For $e^{-\pi Q^2} \ll 2\pi^2 Q^3 T \ll 1$, we get $Z \approx e^{\pi Q^2} (1 + 2\pi^2 Q^3 T)$, which implies that

$$\langle E \rangle = -\frac{d \ln Z}{d\beta} = \frac{T^2}{Z} \frac{dZ}{dT} \approx 2\pi^2 Q^3 T^2 = (2\pi^2 Q^3 T) T \ll T. \quad (57)$$

Preskill *et al.* [5] argue, “The statistical treatment of the radiation is inappropriate if the ensemble of states from which it is drawn is small. In estimating the size of this ensemble, we should not include the residual entropy at zero temperature, since this is unavailable to the radiation.”

However, contrary to this claim, there are actually about $2\pi^2 Q^3 T e^{\pi Q^2} \gg 1$ excited states of energy $E < T$, which is huge in absolute number. For example, it is larger than a googolplex ($10^{10^{100}}$) if $Q \gtrsim \sqrt{\ln 50/\pi} 10^{50} \approx 8.6 \times 10^{49} \approx 2582 Q_* \approx 9.37 \times 10^{11} M_\odot \approx 53$ days, corresponding to a nearly extreme black hole with the mass of a large galaxy. The true reason why $\langle E \rangle \ll T$ for $T \ll 1/Q^3$ is that the number of ground states is even much higher than the enormous number of excited states with $E < T \ll 1/Q^3$, so that the canonical ensemble is dominated by the huge number of degenerate ground states and has only a tiny probability for excitation.

In particular, there are $d_0 \approx e^{\pi Q^2}$ ground states with $E = 0$ for the density of states given above, giving the total probability to be in one or another of these ground states as $P_{\text{ground}} = d_0/Z \approx 1 - 2\pi^2 Q^3 T$. Then the total probability for all excitations is $P_{\text{excite}} = 1 - P_{\text{ground}} \approx 2\pi^2 Q^3 T \ll 1$. Thus

$$\langle E \rangle = P_{\text{ground}} 0 + P_{\text{excite}} \langle E \text{ if excited} \rangle, \quad (58)$$

combined with the value of $\langle E \rangle$ given by Eq. (57), implies that

$$\langle E \text{ if excited} \rangle = \frac{\langle E \rangle}{P_{\text{excite}}} \approx T, \quad (59)$$

which is precisely what one would expect if there were a very dense uniform distribution of states, when T is greater than the average separation between the levels.

If the lowest excited state is nondegenerate and has $E = \delta E \sim e^{-\pi Q^2}/(2\pi^2 Q^3)$, which is the average separation between energy levels for the density of states given by Eq. (55), and if the next excited state has $E = (1+f)\delta E$, etc., then for $T \ll f\delta E$, $P_{\text{excite}} \approx e^{-\pi Q^2 - \delta E/T}$ and $\langle E \rangle \approx (\delta E)e^{-\pi Q^2 - \delta E/T}$. Thus it is only for $T \lesssim 1/(\delta E)$ that the absolute scarcity of excited states with $E < T$ has a significant effect in changing the form of $\langle E \rangle$; for $1/(\delta E) \ll T \lesssim 1/Q^3$, the number of excited states with $E < T$ is huge in absolute number, and it is only the fact that it is less than the even larger number of degenerate ground states that makes $\langle E \rangle \ll T$.

Thus the semiclassical estimate gives a ground state degeneracy of $d_0 \approx e^{\pi Q^2}$ and then a quasi-continuum with an exponentially large density of states $\rho(E) \approx 2\pi^2 Q^3 e^{\pi Q^2}$ for $E \lesssim 1/Q^3$.

One alternative for the energy states of a nearly extreme charged black hole is that there is a whole series of highly degenerate levels, separated by macroscopic mass gaps. For example, the classical formula for extreme black holes with angular momentum $J = n/2$ for nonnegative integers n is

$$E_n = \sqrt{\frac{1}{2}[Q^2 + \sqrt{Q^4 + 4J^2}]} - Q = Q \left\{ \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{n^2}{Q^4}}} - 1 \right\} = \frac{n^2}{8Q^3} - \frac{5n^4}{128Q^7} + \dots \quad (60)$$

One might postulate that each of these levels has degeneracy

$$d_n \approx e^{S(E,J)} = e^{\pi r_+^2} = e^{\pi(Q+E_n)^2} \approx e^{\pi Q^2} \left(1 + \frac{\pi n^2}{4Q^2} \right), \quad (61)$$

possibly multiplied by some function of Q and n that is not exponentially large in Q^2 .

If $E_n \approx E_1 n^a$ and $d_n \propto n^b$ for some constants a and b , then for $T \equiv 1/\beta \gg E_1$,

$$Z = \sum_{n=0}^{\infty} d_n e^{-\beta E_n} \propto \int_0^{\infty} dn n^b e^{-\beta E_1 n^a} \propto \beta^{-(1+b)/a}, \quad (62)$$

which then implies that

$$\langle E \rangle \approx \frac{1+b}{a}T. \quad (63)$$

For example, the alternative above, in which $E_n \approx n^2/(8Q^3)$ and d_n is approximately independent of n , corresponds to $a = 2$ and $b = 0$ and hence gives $\langle E \rangle \approx T/2$ for $E_1 \approx 1/(8Q^3) \ll T \ll 1/Q$. On the other hand, for $T \ll E_1$, $Z \approx d_0 + d_1 e^{-\beta E_1}$, which implies that $\langle E \rangle \approx (d_1/d_0)e^{-E_1/T}$, which is exponentially suppressed when there are no states in an initial mass gap.

There is an opposite alternative for the energy states of a nearly extreme charged black hole that to me seems more plausible in our real world with broken supersymmetry. This alternative is that there is no huge degeneracy anywhere, even for the ground state, but instead only an exponentially dense quasi-continuum of states.

For example, if one takes the semiclassical estimate above and then changes the number of ground states from the value above of $d_0 \approx e^{\pi Q^2}$ to $d_0 = 1$, then one just subtracts roughly $e^{\pi Q^2} - 1$ from the semiclassical estimate for the partition function. This also reduces by roughly $e^{\pi Q^2} - 1$ the number of states of energy $\leq E$ (for $E \geq 0$; for $E < 0$ there are no states) from the approximate value given by Eq. (54) to what now becomes, after inserting the operator Int to take the integer part of what follows,

$$\begin{aligned} N(E) &\approx \text{Int} \left\{ e^{S_0} [I_0(S(E) - S_0) - 1] + 1 \right\} \\ &= \text{Int} \left\{ e^{\pi Q^2} [I_0(\pi r_+^2(E) - \pi Q^2) - 1] + 1 \right\} \\ &= \text{Int} \left\{ e^{\pi Q^2} [I_0(\pi(Q + E + \sqrt{2QE + E^2})^2 - \pi Q^2) - 1] + 1 \right\} \\ &\approx \text{Int} \left\{ e^{\pi Q^2} [I_0(\sqrt{8\pi^2 Q^3 E}) - 1] + 1 \right\} \\ &\approx \text{Int} \left\{ 2\pi^2 Q^3 e^{\pi Q^2} E + 1 \right\}. \end{aligned} \quad (64)$$

Here the penultimate expression on the right hand side (after the second \approx sign) applies for $E \ll Q$, any nearly extreme charged black hole, and the final expression on the right hand side applies for $E \ll 1/Q^3 \ll 1$, where the density of states is nearly uniform, with average spacing given by Eq. (54).

If the temperature is much greater than this average spacing, $e^{-\pi Q^2}/(2\pi^2 Q^3)$, but much less than the maximum temperature of a Reissner-Nordstrom black hole of fixed charge Q , $1/(6\sqrt{3}\pi Q)$, so that then the canonical ensemble is dominated by a large number of excited states that are all nearly extreme black holes (using a tiny cosmological constant to suppress the contribution of the much larger nearly Schwarzschild-anti-deSitter black holes that would have similar or lower temperatures if the cosmological constant were zero), then the partition function has the form given by subtracting roughly $e^{\pi Q^2}$ from that given by Eq. (50):

$$Z \approx e^{\pi Q^2 + 2\pi^2 Q^3 T} - e^{\pi Q^2}. \quad (65)$$

In this temperature range, one thus gets

$$\begin{aligned} \langle E \rangle &= \frac{T^2}{Z} \frac{dZ}{dT} \approx \frac{2\pi^2 Q^3 T^2}{1 - e^{-2\pi^2 Q^3 T}} \\ &\approx \begin{cases} T & \text{for } e^{-\pi Q^2} \ll 2\pi^2 Q^3 T \ll 1 \\ 2\pi^2 Q^3 T^2 & \text{for } 1 \ll 2\pi^2 Q^3 T \end{cases} \end{aligned} \quad (66)$$

Furthermore, for a fixed temperature in this range (and with a tiny cosmological constant to stabilize the canonical ensemble, as discussed above), the von Neumann entropy of the thermodynamic canonical ensemble with the presently-assumed density of states is

$$\begin{aligned} S &= \ln Z + \frac{\langle E \rangle}{T} \approx \pi Q^2 + 2\pi^2 Q^3 T \left(\frac{2 - e^{-2\pi^2 Q^3 T}}{1 - e^{-2\pi^2 Q^3 T}} \right) + \ln(1 - e^{-2\pi^2 Q^3 T}) \\ &\approx \begin{cases} \pi Q^2 + 1 + \ln(2\pi^2 Q^3 T) & \text{for } e^{-\pi Q^2} \ll 2\pi^2 Q^3 T \ll 1 \\ \pi Q^2 + 4\pi^2 Q^3 T & \text{for } 1 \ll 2\pi^2 Q^3 T \end{cases} \end{aligned} \quad (67)$$

On the other hand, for a temperature comparable to or smaller than the spacing between energy levels, $T \lesssim e^{-\pi Q^2}/(2\pi^2 Q^3)$, the discreteness of the energy levels is important. If the levels are indeed evenly spaced with a separation of $e^{-\pi Q^2}/(2\pi^2 Q^3)$ for $E \lesssim 1/Q^3$, then for all $T \lesssim 1/Q^3$ one gets the thermodynamics of a single harmonic oscillator of frequency given by the energy spacing,

$$Z \approx \left[1 - \exp \left(-\frac{1}{2\pi^2 Q^3 e^{\pi Q^2} T} \right) \right]^{-1}, \quad (68)$$

$$\langle E \rangle \approx \left\{ 2\pi^2 Q^3 e^{\pi Q^2} \left[\exp \left(\frac{1}{2\pi^2 Q^3 e^{\pi Q^2} T} \right) - 1 \right] \right\}^{-1}, \quad (69)$$

$$S \approx \left\{ 2\pi^2 Q^3 e^{\pi Q^2} T \left[\exp \left(\frac{1}{2\pi^2 Q^3 e^{\pi Q^2} T} \right) - 1 \right] \right\}^{-1} - \ln \left[1 - \exp \left(-\frac{1}{2\pi^2 Q^3 e^{\pi Q^2} T} \right) \right]. \quad (70)$$

The second and third of these reduce to the expressions given in Eqs. (66) and (67) for $e^{-\pi Q^2} \ll 2\pi^2 Q^3 T \ll 1$, but for $T \ll e^{-\pi Q^2}/(2\pi^2 Q^3)$, $\langle E \rangle$ and S (as well as $Z - 1$) are exponentially suppressed.

These expressions show that if the ground state of a black hole of fixed charge Q is not degenerate (or has a degeneracy much less than the exponential of one-quarter its area of $4\pi Q^2$), but the density of excited states is roughly that given by the semiclassical estimate, then for temperatures T and excess energies E (i.e., energies above that of the ground state energy, which is roughly Q for the fixed charge Q) greater than about $1/Q^3$, the entropy and other thermodynamic properties are very near what one would calculate for a classical black hole with mass given by the charge plus the expectation value of the excitation energy. However, for temperatures and excess energies less than about $1/Q^3$, then the absence of a huge ground-state degeneracy changes the thermodynamic values from what a classical calculation would indicate. In particular, as one lowers the temperature, the thermodynamic entropy is reduced below one-quarter the area of the corresponding classical Reissner-Nordstrom black hole with the same classical temperature or same energy as that of the thermal expectation value. Indeed, when the temperature and excess energy expectation value are reduced to zero, the entropy reduces to the logarithm of the number of degenerate ground states, which is zero if the ground state is not degenerate.

These deduced thermodynamic properties at very low temperatures and energies are different from what a naïve classical calculation would give, but they seem to be a plausible possibility of what happens for charged gravitational systems. Of course, the simple modification of the semiclassical calculation by dropping the degeneracy of the ground state is also naïve, so its detailed predictions are not likely to be correct (e.g., a fairly uniform density of states all the way down to a nondegenerate ground state), but they seem to be a reasonable zeroth-order guess and plausibly are qualitatively correct.

How might one be able to tell in principle whether the ground state of a charged black hole has a huge degeneracy or not? One way is to examine the time for the hole to become extreme.

If the ground state has a huge degeneracy $\sim e^{\pi Q^2}$, comparable to the number of excited states with excitation energies up to about $1/Q^3$, then after the black hole radiates down to $E \sim 1/Q^3$, a reasonable fraction of the lower-energy states are ground states, so it seems plausible that the hole may go to $E = 0$ in one more transition with reasonable probability. Then the expected time to go to one of the ground states would be comparable to (e.g., say a factor of two greater than) the expected time to go to some energy of the order of $1/Q^3$, which we estimated above to be about $10^4 Q^{19}$ and greater than the expected time to emit a positron for $Q \gtrsim Q_* \sim 4 \times 10^8 M_\odot$, which implies a time equal to or greater than about 10^{837} years for a black hole of charge $Q \gtrsim Q_*$ to become extreme and stop emitting (except for emitting positrons, which we found occurs at a slower rate that is suppressed by an exponential of Q/Q_*).

However, if the ground state of a charged black hole has only a small degeneracy, say smaller by a factor of roughly $e^{\pi Q^2}$, then one might expect that the time to decay through all the excited states and reach one of the small number (e.g., one) of ground states might be larger by this same factor (assuming that Q stayed constant). But a time of order $e^{\pi Q^2}$, whose logarithm grows quadratically with Q , is always larger than the time to emit a positron, which goes as e^{Q/Q_*} , whose logarithm rises only linearly with Q . So if this is indeed the case, one would not expect ever to reach a ground state for nonzero charge; the charge would almost always decrease from any fixed value before one reached the ground state for that charge.

Therefore, to tell whether a charged black hole has a ground state degeneracy $\sim e^{\pi Q^2}$ for $Q \gg Q_* \sim 4 \times 10^8 M_\odot$, just wait several times $10^4 Q^{19} \sim 10^{837} \text{yr} (Q/Q_*)^{19}$ and see whether the black hole has stopped radiating. Unfortunately, this is slightly longer than the time over which a Ph.D. student generally hopes to complete his or her thesis research and graduate.

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